[4] (17 points) a) (12 points)Solve the IBVP

$$u_t = u_{rr} + \frac{1}{r}u_r \ ; \ 0 \le r \le 1$$
$$u(1,t) = 0 \quad t \ge 0 \quad \Rightarrow R(1)T(t) = 0 \Rightarrow R(1) = 0$$
$$u(r,0) = r$$
$u(r,t)$ is bounded

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Let $u(r,t) = R(r)T(t)$ and Sub:

$$RT' = R''T + \frac{1}{r}R'T$$

$$\Rightarrow \frac{T'}{T} = \frac{R'' + \frac{1}{r}R'}{R} = -\lambda$$

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$$\Rightarrow R'' + \frac{1}{r}R' + \lambda R = 0$$

$$\Rightarrow r^2R'' + rR' + \lambda r^2 R = 0$$

1) $\lambda = 0$ : $r^2R'' + rR' = 0 \Rightarrow$ euler equation $x^2 + (1-1)x + 0 = 0$, $x^2 = 0 \Rightarrow x = 0$

$\Rightarrow$ Solution is : $R = ar^0 + b(\ln r)r^0 = a + b\ln r$

$R = a + b\ln r$
Since $u(r,t)$ is bounded, $b$ must be zero because $\ln r$ isn't bounded

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$\Rightarrow R = a$
Also $R(1) = 0 \Rightarrow a = 0 \Rightarrow$ not eigenvalue

2) $\lambda = -w^2$ : $r^2R'' + rR' - w^2r^2R = 0$

$t = wr \Rightarrow \frac{dR}{dr} = \frac{dR}{dt}\frac{dt}{dr} = wR'$

$\frac{d^2R}{dr^2} = \frac{d}{dt}\left(\frac{dR}{dr}\right)\frac{dt}{dr} = w^2R''$

$\Rightarrow t^2R'' + tR' - t^2R = 0$

modified Bessel $\nu = 0$ $\Rightarrow R(t) = AI_0(t) + BK_0(t)$

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$u(r,t)$ is bounded $\Rightarrow B = 0$ since $K_0$ isn't bounded
$R(1) = 0 \Rightarrow A = 0$ since $I_0 \ne 0$
$\Rightarrow$ no eigenvalue

3) $\lambda = w^2$ : $r^2R'' + rR' + w^2r^2R = 0$
Similarly, $t = wr$ yields:

$t^2R'' + tR' + t^2R = 0$

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Bessel Eq. with $\nu = 0 \Rightarrow R(t) = AJ_0(t) + BY_0(t)$
Boundness implies $B = 0$ since $Y_0$ isn't bounded

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$\Rightarrow R(t) = AJ_0(t) \Rightarrow R(wr) = AJ_0(wr)$
$R(1) = 0 \Rightarrow AJ_0(w) = 0 \Rightarrow J_0(w) = 0$ $\rightarrow$ cont.

So $w_n = \mu_n$ which are the zeros of $J_0$

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$\lambda_n = \mu_n^2$

$R_n(r) = J_0(\mu_n r)$

Now sub. $\lambda_n$ in T eq.:

$$T' + \lambda_n T = 0$$

$$T' + \mu_n^2 T = 0 \Rightarrow T'(t) = -\mu_n^2 T$$

$$\Rightarrow \frac{T'}{T} = -\mu_n^2 + C$$

1

$$\Rightarrow \ln T = -\mu_n^2 t$$

$$\Rightarrow T = Ce^{-\mu_n^2 t}$$

1

$$\therefore u(r,t) = \sum_{n=1}^{\infty} c_n e^{-\mu_n^2 t} J_0(\mu_n r)$$

$$u(r,0) = r = \sum_{n=1}^{\infty} c_n J_0(\mu_n r)$$

coefficient of R in the SLP

$$\Rightarrow c_n = \frac{\langle r, J_0(\mu_n r)\rangle}{\langle r, r\rangle} = \frac{\int_0^1 r \cdot r J_0(\mu_n r)\,dr}{\int_0^1 r^2 \cdot r\,dr}$$

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$$= \frac{\int_0^1 r^2 J_0(\mu_n r)\,dr}{\int_0^1 r^3\,dr} = 4\int_0^1 r^2 J_0(\mu_n r)\,dr$$

(b) (5 points) Use the series representation

$$J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{v+2k} \textit{ to show that}$$

$$(x^n J_n(x))' = x^n J_{n-1}(x)$$

$$x^n J_n(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(\cancel{(v)}n+K+1)} \left(\frac{x}{2}\right)^{2K+n} x^n$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K \, x^{2K+2n}}{2^{2K+n} \, K!\,\Gamma(K+n+1)}$$

$$\frac{d}{dx}(x^n J_n(x)) = \sum_{K=0}^{\infty} \frac{(-1)^K \, 2\,(K+n)\, x^{2K+2n-1}}{2^{2K+n}\, K!\,\Gamma(K+n+1)}$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K \cdot 2 \cdot \cancel{(K+n)}\, x^{2K+n-1}\, x^n}{2^{2K+n}\, K!\, \cancel{(K+n)}\,\Gamma(K+n)}$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K \, x^{2K+n-1}}{2^{2K+n-1}\, K!\,\Gamma(K+n)} x^n$$

$$= x^n \underbrace{\sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(n+K)} \left(\frac{x}{2}\right)^{2K+n-1}}_{J_{n-1}}$$

$$= x^n J_{n-1}(x)$$

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